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ADJACENCY MATRICES*

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Abstract. For a graph Γ with vertex set V an algebra of adjacency matrices is defined and viewed as an equivalence relation on $V \times V$ with certain nice properties. This can be used in algorithms to find automorphisms of graphs and isomorphisms between graphs. It also provides intersection numbers independent of the labelling on V which determine the similarity class of the adjacency algebra.

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Introduction. This article has two main objectives. The first is to associate as high a dimensional algebra of $n \times n$ matrices as possible with the adjacency matrix of a labelled graph Γ on n vertices. In this way a set of intersection numbers is obtained which is an invariant for the isomorphism class of Γ. The other aim is to show that these intersection numbers provide a finer decomposition into equivalence classes of graphs than do graph spectra, even with the more general definition given here. It therefore seems likely that nice classification theorems must exist using these numbers, giving more powerful results than from spectra. Indeed the theory of distance transitive graphs illustrates this (see [1]). However, such results are not given here. What is provided is the step from a given graph to a coherent configuration as defined by D. G. Higman [4] and one can then apply his theory. He gives some applications.

The associated algorithm which tests for isomorphism by computing these numbers (implicitly) has order at worst $n^3 \log n$ and can be applied recursively to the subgraphs obtained by deleting vertices until isomorphism is established or confuted. The calculation is then producing generalised intersection numbers corresponding successively to ordered pairs, triples, quadruples, etc., of vertices. This points to the correct generalisation to yield invariants which completely determine the isomorphism class of the graph.

Sections 1 to 3 are definitions and elementary properties. Section 4 starts with a couple of well-known results which can be traced back to Frobenius [3]. From them is deduced that intersection numbers are more discriminating than the spectrum. In § 5 these numbers are shown to be equivalent to knowledge of the regular representation, for which a symmetric definition and an easy method of computation are given. Lastly, in § 6, the names of the labels, hitherto ignored, are traced to ensure that an isomorphism preserves not just the equivalence classes of edges carrying the same label, but also the label itself.

The starting point of this paper was a talk by Charles R. Johnson on a joint work of his with Morris Newman [5]. The author would especially like to thank T. J. Laffey for many helpful conversations during its development.

The intersection numbers are obtained in the following way. Let A be an adjacency matrix of a graph Γ. Any automorphism of Γ acts as a similarity transformation by a permutation matrix on A. Thus such transformations act trivially on the algebra generated by all such A for the given graph. A generic matrix of this algebra can be used to partition the vertex set V of Γ into subsets $V_1, V_2, \ldots, V_t$, with the property that any automorphism of Γ restricted to V_i maps onto V_i. The V_i are unions of orbits under the automorphism group.

This can be expressed abstractly using equivalence relations on $V \times V$: giving a "colour" to each edge and vertex. There is a smallest refinement of this colouring of $V \times V$ with a property corresponding to closure under multiplication of matrices. This is called here the *completion* of the colouring, but is just a *coherent configuration* in Higman's terminology.

The formulae in terms of colours for the product of two matrices in this algebra define the intersection numbers and determine the algebra up to similarity. Thus they are identical for isomorphic graphs and can be used as a test for isomorphism The adjacency algebra defined in this way is larger than the usual one, being generated by all possible adjacency matrices instead of a single 0, 1-matrix. It is big enough to show how closely connected are the ideas of similarity, co-spectrality, and intersection numbers.

Addendum. The author would like to note that associating a coherent configuration with

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a graph is the subject of [10]. This does not seem to be well known despite its reference in [11]. The first few sections here describe the method.

1. Colourings.

DEFINITION 1.1. Let V be a finite set and c an equivalence relation on $V \times V$ with r equivalence classes. Then c is called an *r-colouring* or *colouring* of V and the equivalence classes are called the *colours* of c. The set of such classes will be denoted by c and the class of $(i, j) \in V \times V$ by $c(i, j)$. This should be distinguished from $c((i, j))$, also called the *colour* of (i, j), which is always the image of $c(i, j)$ under an injective map. Elements of V are identified with the diagonal of $V \times V$ and called *vertices*, whilst off-diagonal elements are called *edges*.

For example, let Γ be a graph on V with edge set E. Then Γ yields a 3-colouring of V whose colours are V, E, and $(V \times V)\backslash(E \cup V)$.

DEFINITION 1.2. For $n = |V|$ and a commutative ring R containing the integers $\mathbb{Z}$, let $M_V(R)$ be the set of $n \times n$ matrices with entries in R whose rows and columns are indexed by V. Any injective map $c \rightarrow R$ with $c(i, j) \mapsto c((i, j))$ defines a matrix $A = (a_{ij}) \in M_V(R)$ by $a_{ij} = c((i, j))$. Such a matrix is called an *adjacency matrix* of c. Conversely, given a matrix $A = (a_{ij}) \in M_V(R)$ there is a uniquely determined colouring for which A is an adjacency matrix, namely that given by $c(i, j) = c(k, l) \Leftrightarrow a_{ij} = a_{kl}$ for all $i, j, k, l \in V$. The colouring so obtained is denoted c_A. If the set of distinct entries of A are algebraically independent over $\mathbb{Z}$ (as a subring of R) then A is called a *generic matrix* of the colouring it defines. A set of generic matrices (not necessarily for the same colouring) are called *independent* if the entries in each matrix are distinct from the entries in every other matrix and the set of distinct entries from all the matrices is algebraically independent over $\mathbb{Z}$.

LEMMA 1.3. *Let c, d be colourings of V. Then $c = d$ if, and only if, $c((i, j)) = c((k, l)) \Leftrightarrow d((i, j)) = d((k, l))$ for all $i, j, k, l \in V$.*

Example 1.4. The colourings with adjacency matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad A^T = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

are equal.

DEFINITION 1.5. (i) There is a *partial ordering* $\leq$ of colourings given by

$$c \leq d \text{ if, and only if, } c(i, j) \supseteq d(i, j) \text{ for all } (i, j) \in V \times V.$$

(ii) The *sum* or *join* $c + d$ is defined by

$$(c + d)(i, j) = c(i, j) \cap d(i, j)$$

and the *meet* $c \wedge d$ is defined so that $(c \wedge d)(i, j)$ is the smallest union of colours of c containing (i, j) which is also a union of colours of d.

(iii) The rank of a colouring c is the number $|c|$ of equivalence classes of c. Clearly $1 \leq |c| \leq n^2$ for $n = |V|$.

LEMMA 1.6. (i) The *colourings of V form a lattice under $\leq$ with meet and join as above.*

(ii) *$c + d$ is the least upper bound for c and d. In particular, $c \leq c + d$ and $d \leq c + d$. Also $c + d$ is the colouring defined by the sum of independent generic matrices for c and d. Moreover,* $c + d = c$ *if* $d \leq c$.

(iii) *$c \wedge d$ is the greatest lower bound for c and d. In particular, $c \wedge d \leq c$ and $c \wedge d \leq d$ with $c \wedge d = c$ if $c \leq d$.*

(iv) *The map $c \mapsto |c|$ is order preserving, i. e. $c < d$ implies $|c| < |d|$. Also $|c \wedge d| \leq |c| \leq |c+d| \leq |c|\,|d|$.*

DEFINITION 1.7. (i) There is a unique minimal colouring c_0 corresponding to the zero matrix. This is a 1-colouring with $c_0(i,j) = V \times V$.

(ii) There is a unique maximal colouring c_V which is defined by $c_V(i, j) = \{(i, j)\}$. It has $|V|^2$ colours.

(iii) The *identity colouring* c_I is that which corresponds to the identity matrix. It is a 2-colouring with $c_I(i,i) = V \subseteq V \times V$ and $c_I(i,j) = V \times V \setminus V$ for $i \neq j$.

DEFINITION 1.8. The *transpose colouring* c^T is defined by $c^T(i, j) = c(j, i)^T$ where $S^T = \{(i, j) \mid (j, i) \in S\}$ for any subset S of $V \times V$. A colouring c is called *symmetric* if $c = c^T$ and *totally symmetric* if $c(i, j)^T = c(i, j)$ for all $i, j \in V$. Because $(j, i) \in c(i, j)^T$, c is totally symmetric precisely when $c(i, j) = c(j, i)$ for all $i, j \in V$.

Remark 1.9. Suppose A is a generic matrix for c. Then A^T is a generic matrix for c^T.

Example 1.4 illustrates a symmetric colouring which does *not* arise from a symmetric matrix. The totally symmetric colourings are characterised by having symmetric adjacency matrices, whilst the symmetric colourings are characterised by having their set of adjacency matrices closed under the transpose mapping.

The product cd of two colourings is defined as that obtained from the product of independent generic matrices for c and d. Hence we have the following definition.

DEFINITION 1.10. The *product cd* of two colourings c, d of V is defined by its injective image

$$cd((i, j)) = \{c(i, t) \times d(t, j) \mid t \in V\}$$

or, equivalently,

$$cd((i, j)) = \{(\, c((i, t)), d((t, j))\,) \mid t \in V\}$$

where the elements are counted with appropriate multiplicity. *All such sets from here on will be assumed to have multiplicities attached to their elements, i. e. they are* <u>*multisets*</u> *or* <u>*bags*</u>.

In computations as in Example 1.4 the values $c((i, j))$ are usually integers. Then the product class $cd(i, j)$ consists of those directed edges (i, j) yielding the same $|V|$-tuple of pairs $(c(i, t), d(t, j))$ sorted into order. Thus, if c, d are the colourings in Example 1.4, then $cd(2,3)$ is the set of edges giving the triple (13, 21, 32). If generic matrices are used, one has $x_3y_2+x_1y_3+x_2y_1$ representing this class. Ordering the terms lexicographically and recording only subscripts yields the previous triple.

THEOREM 1.11. (i) The *sum and product operations satisfy the usual associative and distributive axioms of rings. Addition is commutative but multiplication is not commutative if* $|V| > 1$.

(ii) $cc_I \geq c$ *and* $c_I c \geq c$.

(iii) $ce \leq df$ *if* $c \leq d$ *and* $e \leq f$ *for colourings* c, d, e, f.

(iv) $c + d \leq cd$ *if* $c \geq c_I$ *and* $d \geq c_I$.

Proof (i) Generic matrices which determine colourings satisfy the named axioms of ring theory. Hence the colourings themselves satisfy these axioms. For $|V| > 1$ let c be the colouring with generic matrix $A = (a_{ij})$ such that $a_{1j} = x$ and $a_{ij} = y$ for $i \neq 1$. Easily $cc^T \neq c^Tc$ since the former is a 4-colouring and the latter the l-colouring.

(ii) $(k, l) \in cc_I\,(i, j)$ implies

$$\{\ c(k, t) \times c_I(t, l) \mid t \in V\ \} \ = \ \{\ c(i, t) \times c_I(t, j) \mid t \in V\ \}$$

Equating terms which contain the diagonal $V = c_I(t, t)$ gives $c(k, l) = c(i, j)$ and hence $(k, l) \in c(i, j)$. Thus $cc_I(i, j) \subseteq c(i, j)$ and $cc_I \geq c$. By symmetry $c_I c \geq c$.

(iii) $(k, l) \in df(i, j)$ implies $\{\ d(k, t) \times f(t, l) \mid t \in V\ \} = \{\ d(i, t) \times f(t, j) \mid t \in V\ \}$ and hence $\{\ c(k, t) \times e(t, l) \mid t \in V\ \} = \{c(i, t) \times e(t, j) \mid t \in V\}$. Therefore $(k, l) \in ce(i, j)$. Thus $df(i, j) \subseteq ce(i, j)$ and $ce \leq df$, as required.

(iv) From (ii) and (iii) $c \leq cc_I \leq cd$ and $d \leq c_I\, d \leq cd$, giving $c + d \leq cd$.

LEMMA 1.12. (i) $c^{TT} = c$;
(ii) $(cd)^T = d^T c^T$ *and* $(c+d)^T = c^T + d^T$;
(iii) cc^T *and* $c + c^T$ *are symmetric* ;
(iv) $c \le d$ *implies* $c^T \le d^T$ *and vice versa* ;
(v) $|c| = |c^T|$.

PROPOSITION 1.13. *Let* c^r *be the product of c with itself r times for* $r \in \mathbb{Z}$, $r > 0$, *and set* $c^0 = c_I$. *Take* $n = |V| > 1$.

(i) *If* $c \ge c_I$, *then there is a positive integer* $m < n^2$ *such that* $c^m = c^{m+r}$ *for all* $r \ge 0$.

(ii) *For each colouring c there are positive integers m, p bounded by functions of n such that* $c^r = c^{p+r}$ *for all* $r \ge m$.

Proof. (i) By Theorem 1.11(ii) and (iii), $c^r \le c^r c_I \le c^{r+1}$ for all $r \ge 0$. If $c_I = c^0 < c^1 < ... < c^r$ then $2 = |c^0| < |c^1| < ... < |c^r|$ by Lemma 1.6(iv) and so $|c^r| \ge r+1$. Now $|c^r| \le n^2$ yields $r < n^2$. Hence there is a maximal value $r = m$ with this property, i.e. $c^m = c^{m+1}$ which gives $c^m = c^{m+r}$ for all $r \ge 0$.

(ii) This is automatic from the finitude of the number of colourings for fixed n.

DEFINITION 1.14. In Proposition 1.13 the minimal m satisfying (i) is called the *order* of c, and the minimal value of p satisfying (ii) is called the *period* of c.

The *completion* of c is $\bar{c} = (c + c^T + c_I)^{n^2}$ for $n = |V|$, and c is called *complete* if $c = \bar{c}$.

Remarks 1.15. Note that $c + c^T + c_I \ge c_I$. Thus, by Proposition 1.13, its period is 1 and $\bar{c} = (c + c^T + c_I)^m$ where m is the order of $c + c^T + c_I$. In computations $\bar{c}$ is obtained by successively squaring $c + c^T + c_I$. The rth squaring gives $(c + c^T + c_I)^{2^r}$ and so $\bar{c}$ results after at most $\log_2(n^2 - 1)$ steps. The computation terminates when squaring returns the same colouring.

$\bar{c}$ is the maximal colouring obtainable from c using c_I and the operations so far defined because of the next theorem.

THEOREM 1.16. (i) $\bar{c}^2 = \bar{c}$; $\bar{c} + \bar{c} = \bar{c}$; *and* $\bar{c}^T = \bar{c}$,

(ii) *If* $c_1 \le \bar{c}$, $c_2 \le \bar{c}$ *then* $c_1 c_2 \le \bar{c}$, $c_1 + c_2 \le \bar{c}$ *and* $c_1^T \le \bar{c}$,

(iii) *If* $c \le d$ *then* $\bar{c} \le \bar{d}$,

(iv) $\bar{\bar{c}} = \bar{c}$,

(v) $\bar{c} c_I = \bar{c} = c_I \bar{c}$ *and* $\bar{c} \ge c_I$

(vi) *c is complete if, and only if,* $c \ge c_I$, $c^T = c$ *and* $c^2 = c$.

THEOREM 1.17. *Suppose c is the totally symmetric* 2- *or* 3-*colouring of a regular graph with adjacency matrix A and* c_i *is the colouring associated with* A^i. *Then, for* $n = |V|$, $\bar{c} = c_0 + c_1 + ... + c_{n-1}$.

Proof. By Theorem 1.16, $c_i \le \bar{c}$ and therefore $c_p \le \bar{c}$ where $c_p = c_0 + c_1 + ... + c_{n-1}$. Now c has generic matrix $xI + yJ + zA$ with $JA = AJ = dJ$ for some $d \in \mathbb{Z}$ and $J^2 = nJ$. So any polynomial in I, J, A is a linear combination of $A^0, A^1, ..., A^{n-1}$ and J by the Cayley-Hamilton theorem. Since $\bar{c} = (c + c_I)^i$ for i large enough, $\bar{c}$ has an adjacency matrix of this form and $\bar{c} \le c_p$, giving $\bar{c} = c_p$.

Remark 1.18. Complete colourings are the same as coherent configurations in the sense of D. G. Higman [4]. The intersection numbers he has are just the multiplicities of the various terms in each entry of a product of two independent generic matrices. Thus completion provides a natural and easy way of associating a coherent configuration with any graph. The completion $\bar{c}$ is the minimal coherent configuration which is a refinement of c. If $\bar{c}$ is totally symmetric then it is an association scheme in the sense of Bose and Shimamoto [2]. If c is obtained from a strongly regular graph, then $\bar{c} = c$ (see J. J. Seidel [8]).

2. Automorphisms.

DEFINITION 2.1. Let S_V denote the group of permutations of V. S_V acts naturally on $V \times V$ by $\sigma(i, j) = (\sigma i, \sigma j)$. Thus σT is well-defined for subsets T of $V \times V$ and $\sigma \in S_V$. In particular, a colouring c with classes $c_1, c_2, ..., c_r$ yields a colouring σc with classes $\sigma c_1, \sigma c_2, ..., \sigma c_r$ where $\sigma c_\kappa = \{(\sigma i, \sigma j) \mid (i, j) \in c_\kappa\}$. The (*strict*) *automorphism group* Aut*c of a colouring c is the subgroup of S_V consisting of permutations which leave the colours fixed, i.e.,

$$\sigma \in \text{Aut}^*\, c \Leftrightarrow \sigma c_i = c_i \quad \text{for each colour } c_i \text{ of } c.$$

Of less interest here is the group Aut $c = \{\sigma \in S_V \mid \sigma c = c\}$ which may include automorphisms which permute the colours nontrivially. For a matrix $A = (a_{ij})$ with associated colouring c, $\sigma A = (\sigma a_{ij})$ is associated with σc and so has entries $\sigma a_{ij} = a_{\sigma^{-1}i,\sigma^{-1}j}$. Then, obviously, Aut* $c = \{\sigma \in S_V \mid \sigma A = A\}$ whilst Aut c consists of those σ for which σA is also an adjacency matrix of c.

LEMMA 2.2. (i) $\sigma c+\sigma d = \sigma(c+d)$; $(\sigma c)(\sigma d) = \sigma(cd)$; $\sigma(c^T) = (\sigma c)^T$ *for* $\sigma \in S_V$;

(ii) Aut* $(c + d)$ = Aut* $c \cap$ *Aut** d ;

(iii) Aut* (cd) = Aut* c $\cap$ Aut* d *if* $c \geq c_I$ *and* $d \geq c_I$;

(iv) *Aut** (c^T)=*Aut** c;

(v) $c \leq d$ *implies* Aut* $c \supseteq$ Aut* d.

Proof. (i), (iv) and (v) are clear.

(ii) If A, B are independent generic matrices for c, d then

$$\sigma \in \text{Aut}^*\,(c+d) \Leftrightarrow \sigma(A+B) = A + B \Leftrightarrow (\sigma A = A \text{ and } \sigma B = B) \Leftrightarrow \sigma \in \text{Aut}^*\, c \cap Aut^*\, d.$$

(iii) Here $\sigma \in$ Aut* $c \cap$ Aut* d implies $\sigma(AB) = (\sigma A)(\sigma B) = AB$ and so $\sigma \in$ Aut* cd. Thus, Aut* $c \cap$ Aut* $d \subseteq$ Aut* cd without restriction. Assuming (v) and using (ii) with Theorem 1.11(iv) gives Aut* $c \cap$ Aut* d = Aut* $(c + d) \supseteq$ Aut* cd and so equality must hold.

THEOREM 2.3. Aut* c = Aut* $(c + c^T + c_I)$ = Aut* $\bar{c}$.

3. Complete colourings.

LEMMA 3.1. *Suppose c is complete*.

(i) $c(i, j) \neq c(k, l)$ *if* $\delta_{ij} \neq \delta_{kl}$ (*Kronecker delta*).

(ii) *If* $c(i, j) = c(k, l)$ *then there is a permutation* $\sigma \in S_V$ *with* $c(i, t) = c(k, \sigma t)$ *and* $c(t, j) = c(\sigma t, l)$ *for all* $t \in V$.

(iii) *If* $c(i, j) = c(k, l)$ *then* $c(i, i) = c(k, k)$ *and* $c(j, j) = c(l, l)$.

Proof. (i) is immediate from $c \geq c_I$. Using the definition of product and $c^2 = c$ gives $\{c(i, t) \times c(t, j) \mid t \in V\} = c^2((i, j)) = c^2((k, l)) = \{c(k, t) \times c(t, l) \mid t \in V\}$. Any bijection between these two bags which preserves colours determines a suitable $\sigma \in S_V$ in (ii). In particular, restricting σ to diagonal classes yields (iii).

THEOREM 3.2 [3, §2.10]. *If* $V = V_1 \mathbin{\dot\cup} V_2 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} V_t$ *is the partition of V induced by the diagonal classes of a complete colouring c then each block* $V_i \times V_j$ *is a union of colours of c*.

COROLLARY 3.3. *With the hypotheses and notation of Theorem* 3.2, *the permutation* $\sigma \in S_V$ *in Lemma* 3.1(ii) *satisfies* $\sigma V_i = V_i$ *for each i*.

COROLLARY 3.4. *Suppose* V_1 *and* V_2 *are diagonal classes* (*possibly equal*) *for a complete colouring c*. *Then* $\{c(i, t) \mid t \in V_2\}$ *and* $\{c(t, j) \mid t \in V_1\}$ *are independent of* $i \in V_1$ *and* $j \in V_2$ *respectively*. *The multiplicities of a colour* c_k *in* $i \times V_2$ *and* $V_1 \times j$ *are related by*

$$|c_k \cap (i \times V_2)|\,|V_1| = |c_k \cap (V_1 \times j)|\,|V_2|.$$

If $c_k \subseteq V_1 \times V_2$ *then* $|V_1|$ *and* $|V_2|$ *divide* $|c_k|$.

Proof. For $i, i' \in V_1$, $c(i, i) = c(i', i')$. So, by Lemma 3.1, there is a $\sigma \in S_V$ with $c(i, t) = c(i', \sigma t)$ for all $t \in V$. By Corollary 3.3, σ restricts to $\sigma_2 : V_2 \to V_2$. Hence $\{c(i, t) \mid t \in V_2\}$ is independent of $i \in V_1$. Independence for the second set follows similarly or by applying the transpose. This immediately gives the equation relating multiplicities, both sides having cardinality $|c_k \cap V_1 \times V_2|$. The last part is now clear.

THEOREM 3.5. The *restriction* c_i *of a complete colouring* c *to* $V_i \times V_i$ *for a diagonal class* V_i *of* c *is a complete colouring with one diagonal class.*

Proof. Clearly $c_i \geq c_I$ and $c_i = c_i^T$ because these properties hold for c. Suppose $c_i(j, k) = c_i(r, s)$. Then $c(j, k) = c(r, s)$ and by Corollary 3.3 the permutation $\sigma \in S_V$ defined in Lemma 3.1 restricts to a map $\sigma_i : V_i \to V_i$ such that $c(j, t) = c(r, \sigma_i t)$ and $c(t, k) = c(\sigma_i t, s)$ for $t \in V_i$. So $c_i^2((j, k)) = \{c(j, t) \times c(t, k) \mid t \in V_i\} = \{c(r, t) \times c(t, s) \mid t \in V_i\} = c_i^2((r, s))$. This means $c_i^2 \leq c_i$ and hence $c_i = c_i^2$. Thus c_i is complete.

Remark 3.6 [3, §8]. In the same way a complete colouring restricts to a complete colouring on any union of its diagonal classes.

DEFINITION 3.7. The number of colours on the diagonal of a colouring $c \geq c_I$ is denoted $\|c\|$. A complete colouring is called *regular* if $\|c\| = 1$ (“homogeneous” in the terminology of Higman).

Remark 3.8. $\|c\|^2 \leq |c|$ for complete colourings.

4. Adjacency algebras and determinants. If we regard a matrix in $M_V(R)$ as a map $V \times V \to R$ in the obvious way, then the adjacency matrices of a colouring c are the maps $\phi : V \times V \to R$ for which every $\phi^{-1}(r)$, $r \in R$, is either the empty set or a colour of c. The adjacency matrices for all colourings $d \leq c$ are the maps $\phi : V \times V \to R$ which are constant on each colour of c, that is, $\phi^{-1}(r)$ is a union of colours of c for all $r \in R$. Such matrices form a free R-module $M_c = M_c(R)$ of rank $|c|$. Certainly $I \in M_c$ if, and only if, $c \geq c_I$. Indeed, $c \leq d$ if, and only if, $M_c \subseteq M_d$. The most important observation is that M_c *is a ring if* c *is complete*. When R is a field and c is complete M_c is therefore an algebra. $M_{\bar{c}}$ is the *adjacency ring* (or algebra) over R of the colouring c.

THEOREM 4.1 (see e.g. Higman [4]). *For a subfield* K *of the complex numbers* $\mathbf{C}$ *and a complete colouring* c *the adjacency algebra* $M_c(K)$ *is semi-simple*.

For the rest of this section take $R = \mathbf{C}$. Since the only division ring over $\mathbf{C}$ is $\mathbf{C}$ itself, Wedderburn’s theorem says that for the decomposition $1 = \sum_{i=1}^m \varepsilon_i$ of 1 into minimal central orthogonal idempotents and $M_i = M_c\varepsilon_i$ there is a decomposition $M_c = \oplus_{i=1}^m M_i$ of M_c into a direct sum of full matrix algebras M_i over $\mathbf{C}$. If M_i consists of $e_i \times e_i$ matrices, then the minimal irreducible left (or right) M_i-modules have dimension e_i and character ζ_i, say. The vector space $\mathbf{C}^V$ on which M_c acts decomposes as $\mathbf{C}^V = \oplus_{i=1}^m \varepsilon_i \mathbf{C}^V$ where $\varepsilon_i \mathbf{C}^V$ is a direct sum of, say, z_i copies of the irreducible M_i-module with character ζ_i. If $\mathbf{C}^V$ has character ζ then $\zeta = \sum_{i=1}^m z_i\zeta_i$ and equating degrees gives $n = |V| = \sum_{i=1}^m z_i e_i$ and $|c| = \sum_{i=1}^m e_i^2$. Clearly $z_i \geq 1$ for each i since the representation of M_c in $M_V(\mathbf{C})$ is faithful.

By the Noether-Skolem theorem there is an invertible matrix $U \in M_V(\mathbf{C})$ such that for all $A \in M_c$,

$$U^{-1}AU = \mathrm{diag}(D_1(A), \ldots, \underbrace{D_i(A), \ldots, D_i(A)}_{\text{multiplicity } z_i}, \ldots, D_m(A))$$

is a block diagonal matrix with $D_i(A)$ affording $\zeta_i(A)$.

For generic A, det $D_i(A)$ is irreducible as follows. Since $D_i(A) = (d_{rs})$ is generic for M_i every entry is distinct and independent of the others. Let det $D_i(A) = fg$ be a nontrivial factorisation and $x = d_{11}$. Without loss of generality $\deg_x f = 1$ and $\deg_x g = 0$. Choose entry y with $\deg_y g = 1$ and $\deg_y f = 0$. As fg contains a term which is a multiple of xy we may assume $y = d_{22}$ by row and column interchanges. Take $d_{12} = d_{21} = 1$, $d_{rs} = O$ otherwise for $r \neq s$, and $d_{rr} = 1$ for $r > 2$. Then det $D_i(A)$ specialises to $xy - 1$ which fails to factorise in the required way. So det $D_i(A)$ is irreducible.

Thus, if A is generic then $\prod_{i=1}^{m}$ det $D_i(A)^{z_i}$ is the factorization of det A into its irreducible factors. Hence det A determines the e_i and z_i (≥ 1) uniquely. They in turn determine $M_c \subseteq M_V(\mathbf{C})$ up to similarity.

Conversely, to obtain a determinant for a given similarity class, pick a matrix representation containing a generic matrix whose distinct entries are linearly independent and which generates the algebra.

THEOREM 4.2. *For a complete colouring c, $M_c(\mathbf{C})$ is determined up to similarity by the determinant of a generic matrix, and conversely.*

Warning 4.3. R. Mathon [6] has some regular graphs on 25 vertices which yield complete 3-colourings that are not isomorphic but have similar adjacency algebras over $\mathbf{C}$. These also appear in [10] and seem to have been computed independently by several people.

Consider next maps $A: c \to \mathbf{C} : c_i \mapsto a_i$ from the colours c_i ($1 \leq i \leq r$) of c into $\mathbf{C}$. Let $\mathbf{C}^c$ denote the set of such maps. If for each $(i, j) \in V \times V$ we are given k such that $c(i, j) = c_k$ then the structure of A as an adjacency matrix is given by $a_{ij} = a_k$ and we obtain a map det $c : \mathbf{C}^c \to \mathbf{C} : A \mapsto$ det (a_{ij}). Clearly, from Theorem 4.2:

COROLLARY 4.4. *For a complete colouring c,* det *c determines the adjacency algebra $M_c(\mathbf{C})$ up to similarity, and conversely.*

The maps in $\mathbf{C}^c$ form an algebra (the regular representation) isomorphic to $M_c(\mathbf{C})$ under the operations induced by the map $A = (a_{ij}) \mapsto A_c$ where $A_c : c(i, j) \mapsto a_{ij}$.

THEOREM 4.5. *Suppose the partition of c into diagonal and off-diagonal colours is given for a complete colouring c. Then $M_c(\mathbf{C})$ is determined up to similarity by multiplication in $\mathbf{C}^c$ defined on its natural basis.*

Proof. By Corollary 4.4 it suffices to reconstruct det c. Multiplication can be described giving the intersection numbers n_{ijk} such that $A_cB_c = F_c \in \mathbf{C}^c$ satisfies $f_j = \sum_{i,k} n_{ijk} a_i b_k$. If $c_i = V_i$ is a diagonal colour then a colour c_j belongs to the block $V_i \times V_i$ if, and only if, $n_{ijj} = n_{jji} = 1$. So $|c_i| = \sum'_{j,k} n_{jik}$ can be found where the sum is restricted to j with $c_j \subseteq V_i \times V_i$. Let $A_c \in \mathbf{C}^c$ and compute $(A^i)_c$ for $i \in \mathbb{N}$. Then each trace Tr (A^i) can be calculated using Tr $A = \sum'_i |c_i| a_i$ where the sum is over diagonal classes. Newton's formulae then yield det A and we obtain det c.

This theorem is implicit in Higman [**4**, §5]. There is a partial converse to the above which is given in [9].

Remark 4.6. Det c provides the spectrum of a graph, and, by virtue of the proof of 4.5, it follows that the intersection numbers determine equivalence classes of graphs which are at least as fine as those given by the spectrum.

5. The regular representation of the adjacency algebra. The adjacency ring $M_c = M_c(R)$ of a complete colouring c is the set of maps $V \times V \to R$ which are constant on colours of c with suitable multiplication. This gives the standard representation of M_c as a ring of matrices operating on R^V. The regular representation is given by considering M_c as the set R^c of maps from the set of colours of c to R. It is obtained as a ring of matrices as follows.

DEFINITION 5.1. For a colouring c and $A_V = (a_{ij}) \in M_V(R)$ the standard (i.e. adjacency matrix) representation of $A \in M_c$, define the matrix $A_c = (a_{lm})$ with entries indexed by colours l, m of c by

$$a_{lm} = |l|^{-1/2}\ |m|^{-1/2} \sum_{i,\, j,t\in V} |(t,i)\cap l||(t,\, j)\cap m| a_{ij}\ .$$

These matrices A_c acting on R^c give the *regular representation of* M_c.

Remark 5.2. Higman [4] makes a slightly different definition for complete colourings, namely

$$a'_{lm} = \sum_{i\in V} |(t,i)\cap l| a_{ij}$$

where $(t, j) \in m$. This is independent of the choice of $t, j \in V$ by virtue of Lemma 3.1(ii). Summing over all such (t, j) to incorporate this symmetry yields $a'_{lm} = |m|^{-1}\sum_{t,i,\, j\in V} |(t,i)\cap l||(t,\, j)\cap m| a_{ij}$. Thus

$$a_{lm} = |l|^{-1/2} |m|^{1/2} a'_{lm}\ .$$

In other words, rows and columns have been multiplied by certain factors.

PROPOSITION 5.3. *Let c_R be the colouring defined on a set of $|c|$ vertices by the regular representation of a colouring $c \geq c_I$. Then c_R is symmetric* (*respectively, totally symmetric*) *if, and only if, c is. Also, $c_R \geq c_I$.*

Proof. First observe that if $l \in c$ and d is the diagonal colour in the same row as l then $a_{dl} = |d|^{-1/2}|l|^{-1/2}\sum_{i:(i,i)\in d}\ \sum_{j:(i,\, j)\in l} a_{ij} = ua_{ij}$ for any $(i, j) \in l$ and some constant $u \neq 0$ dependent only on l. Hence the map $(a_{ij}) \mapsto (a_{lm})$ is one-one. It now suffices to notice from the formula that $(a_{ij})^T \mapsto (a_{lm})^T$.

Finally, a_{ll} contains a nonzero multiple of a_{ii} if $i, t \in V$ are chosen with $(t, i) \in l$, but for no i can a_{ii} appear in a_{lm} if $l \neq m$. So $c_R \geq c_I$.

Examples 5.4. The following are generic adjacency matrices paired with their regular matrix images:

$$\begin{bmatrix} a & c & c & b & d & d \\ c & a & c & d & b & d \\ c & c & a & d & d & b \\ b & d & d & a & c & c \\ d & b & d & c & a & c \\ d & d & b & c & c & a \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \sqrt{2}c & \sqrt{2}d \\ b & a & \sqrt{2}d & \sqrt{2}c \\ \sqrt{2}c & \sqrt{2}d & a+c & b+d \\ \sqrt{2}d & \sqrt{2}c & b+d & a+c \end{bmatrix};$$

$$\begin{bmatrix} a & b & d & d \\ b & a & d & d \\ g & g & e & f \\ g & g & f & e \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & b & \sqrt{2}d & 0 & 0 & 0 \\ b & a & \sqrt{2}d & 0 & 0 & 0 \\ \sqrt{2}g & \sqrt{2}g & e+f & 0 & 0 & 0 \\ 0 & 0 & 0 & e & f & \sqrt{2}g \\ 0 & 0 & 0 & f & e & \sqrt{2}g \\ 0 & 0 & 0 & \sqrt{2}d & \sqrt{2}d & a+b \end{bmatrix}.$$

LEMMA 5.5. *If c is complete, the map $(a_{ij}) \mapsto (a_{lm})$ from the standard to the regular representation is an R-module ring monomorphism.*

Proof. The property for addition is clear. Suppose (a_{ij}) and (a'_{ij}) are two adjacency matrices with images (a_{lm}) and (a'_{lm}). Using the formula in Remark 5.2, $(b_{lm}) = (a_{lm})(a'_{lm})$ has

$b_{lm} = \sum_{n\in c} a_{l,n}a'_{nm} =$

$$\sum_{n\in c}\{|l|^{-1/2}|n|^{-1/2}\sum_{i\in V}|(t,i)\cap l|a_{ij}\}\{|m|^{-1/2}|n|^{-1/2}\sum_{k\in V}|(t,k)\cap m|a'_{jk}\}$$

where $(t, j) \in n$. Summing over all $(t, j) \in n$ and all $n \in c$ yields

$$b_{lm} = |l|^{-1/2}|m|^{-1/2}\sum_{i,k,t\in V}|(t,i)\cap l||(t,k)\cap m|\sum_{j\in V} a_{ij}a'_{jk}$$

which is the lm-entry of the image of $(a_{ij})(a'_{ij})$. As in Proposition 5.3, the map is one-one.

THEOREM 5.6. *Let $V_1, V_2, \ldots, V_t$ be the diagonal colours of a complete colouring c. Suppose n_i is the number of colours in $V_i \times V$, so that $|c| = \sum_{i=1}^{t} n_i$. Then the matrices giving the regular representation of $M_c(\mathbf{C})$ are block diagonal with blocks of size $n_i \times n_i$ for $1 \le i \le t$.*

Proof. Suppose $l, m \in c$ with $l \subseteq V_i \times V$. If $m \subseteq V_i \times V$ then $a_{lm} = 0$ because $|(t, i) \cap l| = 0$ whenever $(t, j) \in m$. The closure under the transpose map described in Proposition 5.3 ensures that $a_{ml} = 0$ also. This establishes the block diagonal nature of the matrices, each block being indexed by the n_i colours in $V_i \times V$ for its rows and columns.

Any map $f: V \to W$ of finite sets can be used to obtain a colouring on fV from a colouring on V. In terms of graphs the map f replaces each set $f^{-1}(w)$ of vertices in V by a single vertex $w \in fV$. In practice, f can be viewed as an equivalence relation on V which identifies various vertices.

DEFINITION 5.7. (i) For subsets S, T of V we define $c(S, T) = \{c(s, t) \mid s \in S, t \in T\}$, counting each $c(s, t)$ with the appropriate multiplicity.

(ii) If $f: V \to W$ is a map of finite sets and c a colouring on V then fc is the colouring on fV defined by $fc((i, j)) = c(f^{-1}i, f^{-1}j)$.

(iii) In case f is written as an equivalence relation ~ on V (mapping V to $\tilde{V}$) we write $\tilde{c}$ for the colouring fc on $\tilde{V}$.

LEMMA 5.8. *If $A = (a_{ij})$ is a generic matrix for the colouring c on V and ~ is an equivalence relation on V then $\tilde{c}$ has adjacency matrix $\tilde{A}$ with entries*

$$\tilde{a}_{uv} = |u|^{-1/2}|v|^{-1/2}\sum_{i\in u}\sum_{j\in v} a_{ij} \quad \text{for } u, v \in \tilde{V}.$$

Note, however, that $\tilde{A}$ need not be generic for $\tilde{c}$.

Proof. Put $a_{uv} = \sum_{i\in u}\sum_{j\in v} a_{ij}$ for $u, v \in \tilde{V}$. Then (a_{uv}) is an adjacency matrix for $\tilde{c}$. For any linear function $f = \sum_{i,j\in V} \lambda_{ij}a_{ij}$ of the a_{ij}'s let $\| f \| = \sum_{i,j\in V} |\lambda_{ij}|$. Then $\|a_{uv}\| = |u||v|$ and $\|\tilde{a}_{uv}\| = |u|^{1/2}|v|^{1/2}$. Hence $a_{uv} = a_{xy}$ if, and only if, $\tilde{a}_{uv} = \tilde{a}_{xy}$. So $(\tilde{a}_{uv})$ is also an adjacency matrix.

THEOREM 5.9. *Define an equivalence relation ~ on V by $i \sim j$ if, and only if, $c(1, i) = c(1, j)$ where c is a complete colouring. Let $A \mapsto \tilde{A}$ be the map $M_c(R) \to M_{\tilde{c}}(R)$ given in Lemma 5.8. Then $\tilde{A}$ is the first block of A_c in the regular representation when the indices are paired $c(1, i)$ with $\tilde{i}$.*

Proof. Let (a_{ij}) be an adjacency matrix for c, $(\tilde{a}_{\tilde{i}\tilde{j}})$ the image under ~ and (a_{lm}) the first block of the regular matrix.

Write $\tilde{i}$ instead of $c(1, i)$ to index the regular matrix block. So

$$a_{\tilde{i}\tilde{j}} = |c(1,j)|^{1/2}|c(1,i)|^{-1/2}\sum_{i\in\tilde{i},j\in\tilde{j}} a_{ij}$$

$$= \quad |\tilde{j}|^{-1} |c(1,j)|^{1/2} |c(1,i)|^{-1/2} \sum_{i\in\tilde{\imath},j\in\tilde{j}} a_{ij} \;=\; \tilde{a}_{\tilde{\imath}\tilde{j}}$$

since $|V_1|\,|\tilde{\imath}| = |c(1,i)|$ where V_1 is the first diagonal colour.

Remarks 5.10. Naturally, Theorem 5.9 is the fastest way to obtain the regular representation. Moreover, this representation is independent of the vertex numbering. By the definitions, it is entirely determined by the intersection numbers, and conversely.

6. Isomorphisms.

DEFINITION 6.1. Let c and d be colourings on V and W respectively. An *isomorphism* from c to d is a bijection $f : V \to W$ such that $fc = d$ in the notation of Definition 5.7. If, in addition, $\phi : c \to d$ is a bijection between the colours of c and d then f is cal1ed a *ϕ-isomorphism* if f induces ϕ on the colours. In particular, if $V = W$ and $c = d$ then an isomorphism is an automorphism and vice-versa; and when ϕ is the identity, then a ϕ-isomorphism is just a strict automorphism. In general, f will map the diagonal colours of c onto the diagonal colours of d and applying the transpose to colours commutes with the map f induces on colours. We will require ϕ to have these properties.

If c and d arise from two graphs then ϕ is usually the map which matches properties of one graph with those of the other. Then the existence of a ϕ-isomorphism from c to d is equivalent to the graphs being isomorphic. By viewing $f : V \to W$ as a re-naming of subscripts, we have (cf. Lemma 2.2(i)) the next lemma.

LEMMA 6.2. *Let $f : V \to W$ be injective and c, d colourings on V. Then*

(i) *a generic matrix for c is generic for fc* ;

(ii) $f(cd) = f(c)f(d)$; $f(c+d) = f(c) + f(d)$; $f(c^T) = (fc)^T$;

(iii) $f(\overline{c}) = \overline{fc}$;

DEFINITION 6.3. Let c, d be symmetric colourings $\geq c_I$. Suppose $\phi : c \to d$ is a bijection of colours which restricts to a bijection between the diagonal colours and which commutes with the transpose map. There is an induced R-module isomorphism $\Phi : R^c \to R^d$ of regular representations. If Φ commutes with multiplication, then it extends to a map $\Phi^2 : R^{c^2} \to R^{d^2}$: $AB \to \Phi(A)\Phi(B)$ for $A, B \in R^c$. This yields a bijection $\phi^2 : c^2 \to d^2$. Equivalently, if for all i, $j \in V$ there are $r, s \in W$ with $\{\phi c(i,t) \times \phi c(t,j) \mid t \in V\} = \{d(r,t) \times d(t,s) \mid t \in W\}$ then ϕ has a natural refinement to a bijection $\phi^2 : c^2 \to d^2$, namely $\phi^2 c^2(i,j) = d^2(r,s)$. Note, however, that ϕ^2 can be found from the multiplications $R^c \times R^c \to R^{c^2}$ and $R^d \times R^d \to R^{d^2}$ without referring back to the standard representation. In the same way, it may be possible to define $\phi^r : c^r \to d^r$ for all $r > 0$. Then iteratively one obtains a bijection $\overline{\phi} : \overline{c} \to \overline{d}$ inducing $\overline{\Phi}$: $R^{\overline{c}} \to R^{\overline{d}}$. If this is an R-ring isomorphism, i.e. preserves multiplication, or equivalently, $\overline{\phi}^2 = \overline{\phi}$, then we say $\overline{\phi}$ is *complete*.

There is an obvious correspondence between adjacency matrices $A = (a_{ij}) \in M_c(R)$ and $B = (b_{rs}) \in M_d(R)$ when there is a bijection $\phi : c \to d$ namely that with $a_{ij} = b_{rs}$ whenever $\phi c(i,j) = d(r,s)$. Again, let Φ: $M_c(R) \to M_d(R)$ denote the map. We say c and d are *cospectral* (under ϕ) if, and only if, $\det A = \det \Phi A$ for all $A \in M_c(R)$, (i.e. if, and only if, $\det c = \det d_0\Phi$) and ϕ gives a bijection between diagonal colours.

THEOREM 6.4. *Suppose $f : V \to W$ is a ϕ-isomorphism of the colourings c, d. Then there is a natural way of refining ϕ to a complete bijection $\overline{\phi} : \overline{c} \to \overline{d}$ independently of f so that f is a $\overline{\phi}$-isomorphism from $\overline{c}$ to $\overline{d}$.*

Proof. $\overline{c}$ and $\overline{d}$ are isomorphic under f by 6.2(iii). *Since* $f(AB) = f(A)f(B)$ for all A, $B \in M_c(R)$, $\phi^2 : c^2 \to d^2$ may be defined by $\phi^2 c^2(i,j) = d^2(fi, fj) \equiv \{d(fi,t) \times d(t,fj) \mid t \in W\} =$

$\{\phi c(i, t) \times \phi c(t, j) \mid t \in V\}$. So $\overline{\phi}$ is obtained by iteration, and it is complete.

THEOREM 6.5. *Suppose ϕ is a bijection between the colours of c and d, and ϕ can be refined to a complete bijection $\overline{\phi} : \overline{c} \to \overline{d}$ of colours. Then $M_{\overline{c}}(\mathbf{C})$ and $M_{\overline{d}}(\mathbf{C})$ are similar, and c and d are cospectral (under ϕ). If $V_1, ..., V_r$ are the diagonal colours of $\overline{c}$ and $\phi V_i = V_i$ for all i then there is a unitary matrix U, necessarily block diagonal under the partition given by the V_i's, such that $U\Phi A = AU$ for all adjacency matrices A of $\overline{c}$. Here A and ΦA have identical characteristic polynomials. There is also a block diagonal matrix $U_{\mathbb{Q}}$ with rational entries such that $U_{\mathbb{Q}}\Phi A = AU_{\mathbb{Q}}$ for all such matrices A. Moreover, U and $U_{\mathbb{Q}}$ may be chosen to have row and column sums equal to* 1.

Proof. The regular representations are identical except for the indexing by $\overline{c}$ or $\overline{d}$. Now apply Theorems 4.2 and 4.5.

There is a unitary matrix $U \in M_v(\mathbf{C})$ independent of the choice of A such that $\Phi A = U^{-1}AU$. Decomposing into blocks under the diagonal colours gives $\sum_t U_{it}\Phi A_{tj} = \sum_t A_{it}U_{tj}$. If A is a generic matrix whose elements are independent of those in U, then equating terms from the block A_{ij} yields $U_{ii}\Phi A_{ij} = A_{ij}U_{jj}$ and $U_{it} = 0$ for $t \neq i$. Hence U is block diagonal with unitary diagonal blocks.

The matrix $U_{\mathbb{Q}}$ is obtained by observing that without loss of generality U has algebraic number entries and then summing $U\Phi A = AU$ over all conjugates. $A = J = \Phi A$ gives the row and column sum property.

ALGORITHM 6.6. The graph isomorphism problem is that of finding a permutation matrix U such that $U\Phi A = AU$ for corresponding adjacency matrices A, ΦA of two graphs. This has been translated into finding a permutation f: $V \to W$ of the vertex sets which is a ϕ-isomorphism of the appropriate colourings c, d. By Theorem 6.4 there must be a complete bijection $\overline{\phi} : \overline{c} \to \overline{d}$. A basic check for isomorphism therefore involves iteratively forming c^{2^i}, d^{2^i} and ϕ^{2^i} to obtain $\overline{\phi} : \overline{c} \to \overline{d}$. This establishes that the regular representations are the same so that the standard representations by adjacency algebras are similar and the graphs co-spectral. The partitioning of the vertices via the diagonal colours serves to restrict the possible permutations if the graphs are isomorphic and standard techniques (see [7]) enable a tree of completions to be used to yield isomorphisms.

To construct the completions for two graphs and the map between their colours, represent the graphs by adjacency matrices with integer entries that are equal for edges if and only if they have identical labels in the graphs. These entries can be chosen in the range 1 to n^2 for $n = |V|$. If this can be done in $O(n^3)$ time then the 2 $\log_2 n$ squarings lead to an $O(n^3 \log n)$ time bound on completion, assuming that integers in the range 1 .. n^2 can be accessed and compared in unit time. First of all, observe that even bubble sort will sort the elements of each row into order in $O(n^3)$ time, providing a permutation to reorder the elements as they appear in the row, and information about repeated elements. The same applies to columns.

Each of the n^2 elements of the square is given by a formal dot product of a row with a column. The information about how to sort both row and column must be combined to sort the n-tuple in linear time. For each distinct value in the row we have a series of adjacent spaces in the final sorted n-tuple into which terms containing that value will be placed. Assign a pointer for each such value, setting it to the first such place which is empty. Now use the column order to take each term in turn, placing it according to the corresponding row pointer, and incrementing that pointer. This sorts the n-tuple in $O(n)$ time.

The other part of the squaring procedure involves renumbering entries to obtain new numbers which are equal if and only if the corresponding sorted n-tuples are equal. This is done by renumbering using the first term, then taking the new numbering with the second term, and so on. Thus, all n-tuples must be sorted first, requiring $O(n^3)$ space to be available. Each n-tuple is represented by a vector of $2n$ integers in the range $1 \,..\, n^2$. It suffices to show how to incorporate the first element of each into the new numbering in $O(n^2)$ time to achieve the $O(n^3)$ time requirement for squaring.

Generally, a unique numbering is obtainable for m ordered pairs of integers in the range $1 \,..\, k$ in $O(\max(k, m))$ time. We apply this to pairs given by the current matrix numbering with the next element in each vector. The numbering is achieved by setting k list head pointers to zero and scanning each pair to set up linked lists connecting pairs with the same initial element; then each list is scanned to form sublists divided according to the second element; finally the lists are scanned again, assigning a new number of each sublist: $O(k+m)$ time.

The above process must be carried out simultaneously on both graphs to ensure common renumberings. If at any point a discrepancy arises – differing multiplicities between the two adjacency matrices – then the graphs cannot be isomorphic and indeed, eventually there are no numbers in common in the completions. If the completions do agree then the graphs are similar if not actually isomorphic.

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