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Brauer's class number relation

by

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The main part of this paper proves R. Brauer's class number relation [1] in a shorter and more natural way. Consequently it is possible to obtain Stark's generalization [8] with no extra effort and to observe that the theorem may be applied using only the units of the occurring fields. Nehrkorn's conjecture [6] that there exists a corresponding class group isomorphism is also shown to be correct.

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1. Relation theorems. In this first section are derived some general results to describe relations in torsion modules and in torsion-free modules. All the modules concerned will be finitely generated.

Let $\mathfrak{O}$ be a Dedekind domain contained in a field K of characteristic zero and write $\mathfrak{O}_\mathfrak{p} = \{\alpha/\beta \in K \mid \alpha \in \mathfrak{O},\ \beta \in \mathfrak{O} - \mathfrak{p}\}$ for its localisation at the prime ideal $\mathfrak{p}$. Then a $\mathfrak{O}$-lattice M is a finitely generated torsion-free $\mathfrak{O}$-module. M will be identified with its natural embedding in $KM = K \otimes_\mathfrak{O} M$ and $M_\mathfrak{p}$ will be written for $\mathfrak{O}_\mathfrak{p} \otimes_\mathfrak{O} M$.

If M and N are two $\mathfrak{O}$-lattices of $KM = KN$ then the index $[M : N]$ may be defined through the local indices $[M_\mathfrak{p} : N_\mathfrak{p}]$ for the free $\mathfrak{O}_\mathfrak{p}$-modules $M_\mathfrak{p}$ and $N_\mathfrak{p}$. Let $\delta_\mathfrak{p}$ be the determinant of a matrix which describes a basis of $N_\mathfrak{p}$ in terms of one for $M_\mathfrak{p}$. Then $[M_\mathfrak{p} : N_\mathfrak{p}] = \mathfrak{O}_\mathfrak{p}\delta_\mathfrak{p}$ is well-defined and non-zero. By taking free $\mathfrak{O}$-submodules of M and N with the same rank as M and N it is clear that the $\delta_\mathfrak{p}$ can be chosen equal for almost all $\mathfrak{p}$ and that the ratio of two $\delta_\mathfrak{p}$ is always in the field of fractions k of $\mathfrak{O}$. Hence the intersection over all primes $\mathfrak{p}$ which defines the index, viz.

$$[M : N] = \bigcap_{\mathfrak{p}} [M_\mathfrak{p} : N_\mathfrak{p}]$$

is the product of an ideal in $\mathfrak{O}$ and an element of K. If M and N are isomorphic then $[M : N] = \mathfrak{O}\delta$ for the determinant $\delta \in K$ of the corresponding automorphism of KM. Thus for $\mathfrak{O} = \mathbb{Z}$ and $K = \mathbf{C}$ this coincides with the usual definition of the index viewed as an ideal, and when $K = k$ the definition coincides with that of Fröhlich [2]. If K/k is a number field extension with norm N_k^K, $\mathfrak{O}_k$ is the ring of integers of k, and $\mathfrak{a}$, $\mathfrak{b}$ are ideals of K then $[\mathfrak{a}:\mathfrak{b}] = N_k^K(\mathfrak{a}^{-1}\mathfrak{b})$.

Now let G be a finite group. A $\mathfrak{O}[G]$-lattice is just a $\mathfrak{O}$-lattice which is a $\mathfrak{O}[G]$-module.

THEOREM 1.1. *Suppose* $\{e_i\}$ *is a finite set of idempotents in* $k[G]$, χ_i *is the character of* $K[G]e_i$, *and* $\sum a_i\chi_i = 0$ *for* $a_i \in \mathbb{Z}$. *If* M *and* N *are isomorphic* $\mathfrak{O}[G]$*-lattices and* $KM = KN$ *then*

$$\prod_i [M_i : N_i]^{a_i} = \mathfrak{O}$$

where $M_i = M\cap e_iKM$ *and* $N_i = N\cap e_iKN$.

Proof. Any two $K[G]$-modules X and Y and a $K[G]$-automorphism α of X induce an automorphism α_Y of $\mathrm{Hom}_{K[G]}(Y, X)$, namely $\alpha_Y(f) = \alpha \circ f$. Clearly $\det \alpha_Y = \Delta(\alpha, \chi)$ depends only on α and the isomorphism class of Y, which is determined by the character χ of Y. If $Y = Y_1 \oplus Y_2$ then $0 \neq \det \alpha_Y = (\det \alpha_{Y_1})(\det \alpha_{Y_2})$, and so

$$\Delta(\alpha, \chi_1+\chi_2) = \Delta(\alpha, \chi_1)\, \Delta(\alpha, \chi_2).$$

Thus $\chi \mapsto \Delta(\alpha, \chi)$ extends to a homomorphism from the additive group of the virtual characters of G into the multiplicative group of K. In particular,

$$(*) \qquad \sum a_i\chi_i = 0 \quad \Rightarrow \quad \prod \Delta(\alpha, \chi_i)^{a_i} = 1\,.$$

Let e be an idempotent of $K[G]$ and χ the character of $Y = K[G]e$. Then there is a K-isomorphism $\beta : \mathrm{Hom}_{K[G]}(Y, X) \xrightarrow{\sim} eX$ given by $f \mapsto f(e)$ with inverse $x \mapsto (f\colon y \mapsto yx)$. Define α_e as the restriction of α to eX. Then $\alpha_e \circ \beta = \beta \circ \alpha_Y$ from which $\Delta(\alpha, \chi) = \det(\alpha_e)$. If α is chosen so that $\alpha M = N$ then $\mathfrak{O}\det(\alpha_e) = [M_i : N_i]$ and (*) proves the theorem.

Remark (J.-J. Payan). From the local definition of index, the theorem still holds if the $\mathfrak{O}[G]$-lattices M and N are just assumed to be in the same genus, i.e. $M_\mathfrak{p} \cong N_\mathfrak{p}$ for all $\mathfrak{p}$.

THEOREM 1.2. *Let* $S = \{e_i\}$ *be a finite set of idempotents in* $k[G]$ *and* $\mathfrak{O}_S$ *the subring of k generated over* $\mathfrak{O}$ *by* $|G|^{-1}$ *and the coefficients of the* $e_i \in S$. *Suppose* χ_i *is the character of* $k[G]e_i$ *and* $\sum a_i\chi_i = \sum b_i\chi_i$ *for some non-negative integers* a_i *and* b_i. *If* M *is a finite group and a* $\mathfrak{O}_S[G]$*-module then there is a* $\mathfrak{O}_S$*-module isomorphism*

$$\oplus_i \oplus_{j=1}^{a_i} e_iM^{(j)} \cong \oplus_i \oplus_{j=1}^{b_i} e_iM^{(j)} \qquad \textit{for} \quad M^{(j)} \cong M\,.$$

Proof. Again let $M_\mathfrak{p} = \mathfrak{O}_\mathfrak{p}\otimes_\mathfrak{O}M$ for each prime $\mathfrak{p}$ of $\mathfrak{O}$. Then $M_\mathfrak{p}$ is a $\mathfrak{O}_\mathfrak{p}[G]$-module which is trivial for almost all $\mathfrak{p}$ and, in particular, for $\mathfrak{p}$ dividing the ideal $\mathfrak{O}|G|$. As $M \cong \oplus_\mathfrak{p} M_\mathfrak{p}$ we may assume without loss of generality that $M = M_\mathfrak{p}$ for some prime $\mathfrak{p}$ not dividing $\mathfrak{O}|G|$.

Let $N_i = \mathfrak{O}_\mathfrak{p}[G]e_i$. Then there is a $\mathfrak{O}_\mathfrak{p}$-isomorphism $\mathrm{Hom}_{\mathfrak{O}_\mathfrak{p}[G]}(N_i, M) \xrightarrow{\sim} e_iM$ given by $f \mapsto f(e_i)$. Two $\mathfrak{O}_\mathfrak{p}[G]$-lattices N and N', of the same character satisfy $kN \cong kN'$ and therefore the work of Maranda ([5], Theorem 4) shows that $N \cong N'$. Combining these isomorphisms gives

$$\begin{aligned}\oplus_i \oplus_{j=1}^{a_i} e_i M^{(j)} &\cong \mathrm{Hom}(\oplus_i \oplus_{j=1}^{a_i} N_i, M) \\ &\cong \mathrm{Hom}(\oplus_i \oplus_{j=1}^{b_i} N_i, M) \cong \oplus_i \oplus_{j=1}^{b_i} e_i M^{(j)}.\end{aligned}$$

2. Nehrkorn's theorem. Let K/k be a normal extension of algebraic number fields with Galois group G. Suppose 1_H^G is the character on G induced from the unit character on a subgroup H, and for a module X on which G acts let HX be the submodule fixed under H. Write $\widetilde{H}$ for the sum of the elements in H. As usual let us define U to be the group of units in K; W its subgroup of roots of unity; $w(H)$ the order of HW; and $w_2(H)$ the 2-component of $w(H)$.

THEOREM 2.1. *Let $C(HK)$ be the part of the ideal class group of HK formed from classes whose orders are prime to $|G|$. If*

$$\sum_H a(H)\, 1_H^G = \sum_H b(H)\, 1_H^G$$

where $a(H)$ and $b(H)$ are non-negative integers then there is a group isomorphism

$$\oplus_H \oplus_{j=1}^{a(H)} C(HK)^{(j)} \cong \oplus_H \oplus_{j=1}^{b(H)} C(HK)^{(j)} \quad \text{for } C(HK)^{(j)} \cong C(HK).$$

Nehrkorn indicated in [6] that the above result holds but proved it only for K/k abelian. It is immediate from Theorem 1.2 because of the natural isomorphism $C(HK) \cong HC(K)$ and because the character 1_H^G corresponds to the idempotent $\widetilde{H}/|H|$.

LEMMA 2.2. *Suppose M is a finite $\mathbb{Z}[G]$-module fixed by a normal subgroup N over which G is cyclic. If $\sum a(H)1_H^G = \sum b(H)1_H^G$ where $a(H)$ and $b(H)$ are non-negative integers then there is a trivial group isomorphism*

$$\oplus_H \oplus_{j=1}^{a(H)} HM^{(j)} \cong \oplus_H \oplus_{j=1}^{b(H)} HM^{(j)} \qquad \text{for } M^{(j)} \cong M.$$

Proof. From $\sum 1_H^G(g)g = |H|^{-1}\sum g\widetilde{H}g^{-1}$ for both sums over $g \in G$ we deduce that $1_H^G(g\widetilde{N}) = |N| 1_{HN}^G(g)$. Hence $\sum a(H) 1_{HN/N}^{G/N} = \sum_H b(H) 1_{HN/N}^{G/N}$. By Brauer [1], Satz 2, or Rehm [7], Satz 1, this relation is trivial for G/N cyclic. Thus the stated group isomorphism holds trivially as M is a $\mathbb{Z}[G/N]$-module with $HM = (HN/N)M$.

THEOREM 2.3 (Brauer [1], §5). *If* $\sum a(H)1_H^G = 0$ *then*

$$\prod_H w(H)^{a(H)} = \prod_H w_2(H)^{a(H)} .$$

Suppose also that W_2 *is the group of* 2*-power roots of unity in* K *and* $k(W_2)/k$ *is cyclic. Then*

$$\prod_H w_2(H)^{a(H)} = 1 .$$

Proof. Let W_p be the Sylow p-subgroup of W. Then, with the possible exception of $p = 2$, $k(W_p)/k$ is cyclic and the theorem is a direct consequence of Lemma 2.2 on taking orders.

3. Brauer's theorem. With the notation of §2 let us assume also that $k = \mathbb{Q}$; $n(H) = [G{:}H]$ is the degree of HK over k; $r_1(H)$ and $r_2(H)$ are the numbers of real and non-real infinite valuations of HK; $r(H)$ is the rank of HU/HW; $R(H)$ is the regulator and $h(H)$ the class number of HK; and $\delta(H) = 2$ or 1 according as HK is totally complex or not. For some fixed embedding of K into the complex numbers $\mathbf{C}$ let C be the Galois group of the maximal real subfield of K. Thus C is generated by the automorphism γ which induces complex conjugacy on K.

Let L and L^* be $\mathbb{Z}[G]$-lattices which make

$$0 \to \mathbb{Z} \to \mathbb{Z}[G]\tilde{C} \to L \to 0 \quad \text{and} \quad 0 \to L^* \to \mathbb{Z}[G]\tilde{C} \to \mathbb{Z} \to 0$$

exact sequences of left $\mathbb{Z}[G]$-modules. Here the maps from and to $\mathbb{Z}$ are given by $n \mapsto n\tilde{G}$ and $a\tilde{C} \mapsto 1(a)$ respectively for the unit character 1. Specifically, L and L^* will be identified with $\mathbb{Z}[G]\tilde{C}/\mathbb{Z}\tilde{G}$ and $\{a\in \mathbb{Z}[G]\tilde{C} \mid 1(a) = 0\}$. Denote by a bar the natural maps $U \to U/W$ and $\mathbb{Z}[G] \to \mathbb{Z}[G]/\mathbb{Z}\tilde{G}$ and define maps $\lambda : \overline{U} \to \mathbf{C}L$ and $\lambda^* : \overline{U} \to \mathbf{C}L^*$ by

$$\lambda(\overline{\varepsilon}) = \sum_{g\in G} \log\|g^{-1}\varepsilon\|\overline{g} ,$$

$$\lambda^*(\overline{\varepsilon}) = \sum_{g\in G} \log\|g^{-1}\varepsilon\| g \qquad \text{for } \varepsilon \in U$$

where $\| \ \|$ is the absolute value of the chosen embedding of K into $\mathbf{C}$. These are both $\mathbb{Z}[G]$-homomorphisms and they are injections because the ranks of $\lambda(\overline{U})$, $\lambda^*(\overline{U})$, L, L^*, and $\overline{U}$ are all equal by the next theorem and the Dirichlet unit theorem.

THEOREM 3.1. *We have*

$$[HL : \lambda\overline{H}\overline{U}] = \mathbb{Z}n(H)2^{-r_2(H)}R(H)$$

and

$$[HL^* : \lambda^*\overline{H}\overline{U}] = \mathbb{Z}\delta(H)2^{-r_2(H)}R(H) .$$

Proof. Let HgC denote the sum of the distinct elements in $\{hgc \mid h\in H,\ c\in C\}$ and $|HgC|$ the number of such elements. If possible choose

$g_0 \in G$ such that Hg_0C is a single coset of $\tilde{H}$ and otherwise take any g_0. Then $\delta(H) = |Hg_0C|/|H|$ and the elements $HgC - |HgC||Hg_0C|^{-1}Hg_0C$ generate HL^* over $\mathbb{Z}$. For $\varepsilon \in HU$ we have

$$\lambda^*(\bar{\varepsilon}) = \sum \log\|g^{-1}\varepsilon\|HgC = \sum \log\|g^{-1}\varepsilon\|(HgC - |HgC||Hg_0C|^{-1}Hg_0C)$$

for sums over double coset representatives $g \in H\backslash G/C$. Hence

$$[HL^*:\lambda^*\overline{H\bar{U}}] = \mathbb{Z}\delta(H)2^{-r_2(H)}R(H)\,.$$

Now $[HL:\lambda\overline{H\bar{U}}] = [HL:\overline{HL}^*]\,[HL^*:\lambda^*\overline{H\bar{U}}]$ because $CL^* \to C\bar{L}^*$ is an isomorphism. As a basis of HL is given by $\{\overline{Hg\bar{C}}\}$ for $g \in H\backslash G/C$ with $g \notin Hg_0C$ so $[HL:\overline{HL^*}] = \mathbb{Z}\det(a_{g,g'})$ where $a_{g,g'} = \delta_{g,g'}+|HgC|/|Hg_0C|$ for the Kronecker delta δ. But $\sum_g a_{g,g'} = n(H)/\delta(H)$ gives a constant row by which $|HgC|/|Hg_0C|$ may be subtracted from each $a_{g,g'}$. Thus $[HL:\overline{HL^*}] = \mathbb{Z}n(H)/\delta(H)$ as required.

LEMMA 3.2. *If* $\sum a(H)1_H^G = 0$ *then*

$$0 = \sum a(H) = \sum a(H)r_1(H) = \sum a(H)r_2(H) = \sum a(H)r(H) = \sum a(H)n(H)\,.$$

Proof. The sums are the evaluations of the relation at $\tilde{G}/|G|$, γ, $(1-\gamma)/2$, $\tilde{C}/|C|-\tilde{G}/|G|$, and 1 respectively because

$$r_1(H) = |\{g\in G \mid g\gamma g^{-1} \in H\}|/|H| = 1_H^G(\gamma)\,.$$

LEMMA 3.3. *If* $\sum a(H)1_H^G = 0$ *and* $M, M^* \subset \bar{U}$ *are* $\mathbb{Z}[G]$*-isomorphic to* L, L^* *respectively then*

$$\prod R(H)^{-a(H)} = \prod (n(H)[\overline{H\bar{U}}: HM])^{a(H)} = \prod (\delta(H)\,[\overline{H\bar{U}}:HM^*])^{a(H)}.$$

Proof. M and M^* exist because λ and λ^* are injective homomorphisms so that L, L^*, and $\bar{U}$ all have the same character.

Let $\pi = \prod (n(H)[\overline{H\bar{U}}:HM]R(H))^{a(H)}$. By Theorem 3.1

$$\mathbb{Z}\pi = \prod (2^{r_2(H)}[\lambda\overline{H\bar{U}}:\lambda HM]\,[HL:\lambda\overline{H\bar{U}}])^{a(H)}\,.$$

Hence Lemma 3.2 and Theorem 1.1 yield

$$\mathbb{Z}\pi = \prod [HL:\lambda HM]^{a(H)} = \prod [HL:H\lambda M]^{a(H)} = \mathbb{Z}$$

from $L \cong \lambda M$. The other relation holds similarly.

Application of the functional equation to the residue of the zeta function $\zeta_{HK}(s)$ at $s = 1$ gives the well-known result

$$\lim_{s\to 0} s^{-r(H)}\zeta_{HK}(s) = -h(H)R(H)/w(H)\,,$$

while the interpretation of $\zeta_{HK}(s)$ as the Artin L-series $L(s, 1_H^G, K/\mathbb{Q})$ shows that $\sum a(H)1_H^G = 0$ implies $\prod \zeta_{HK}(s)^{a(H)} = 1$. Equating values at $s = 0$ and using Lemma 3.2 yields (Kuroda [3])

$$\prod_H \bigl(h(H)R(H)/w(H)\bigr)^{a(H)} = 1.$$

Comparing this with the limit of $\prod \zeta_{HK}(s)^{a(H)} = 1$ as $s \to 1$ shows that the corresponding product of discriminants is also 1. However, combining it with Lemma 3.3 and Theorem 2.3 immediately provides Brauer's theorem ([1], Satz 4) :

THEOREM 3.4. *If the submodules M and M* of $\overline{U}$ are $\mathbb{Z}[G]$-isomorphic to L and L* respectively and if* $\sum a(H)1_H^G = 0$ *then*

$$\prod_H h(H)^{a(H)} = \prod_H \bigl(n(H)w_2(H)[\overline{H\overline{U}} : HM]\bigr)^{a(H)}$$
$$= \prod_H \bigl(\delta(H)w_2(H)[\overline{H\overline{U}} : HM^*]\bigr)^{a(H)}.$$

Remark 3.5. Set $S = \{H \mid a(H) \neq 0\}$ and let U_S be the group generated over $\mathbb{Z}[G]$ by $\{HU \mid H \in S\}$. Then $\overline{U_S}$ may have smaller rank than $\overline{U}$ so that more units need to be calculated to obtain a module M. However, suppose L_S is a $\mathbb{Z}[G]$-module satisfying $HL \subset L_S \subset L$ for all $H \in S$ and $M' \subset \overline{U}$ is the corresponding submodule of M. As $HM = HM'$ for all $H \in S$ we may replace M by M' in the theorem and by Theorem 1.1 the substitution of any module $M_S \subset \overline{U}$ which is $\mathbb{Z}[G]$-isomorphic to L_S is also valid. In particular, the minimal choice of L_S ensures that $M_S \subset \overline{U_S}$. A module M_S^* can be defined analogously. *It is therefore possible to apply Theorem 3.4 when only the units of the occurring subfields are known.*

Remark 3.6. The full extent of Theorem 1.1 has not yet been exploited but we expect that when the value of

$$c(\chi) = \lim_{s\to 0} s^{-r(\chi)} L(s, \chi, K/\mathbb{Q})$$

has been calculated for $r(\chi) = \chi(\tilde{C}/|C| - \tilde{G}/|G|)$ and any character χ then the same techniques will produce a relation similar to Theorem 3.4 (see Lichtenbaum [4]). An intermediate result can be obtained. If the character ρ is irreducible over $\mathbb{Q}$, contains an absolutely irreducible character of degree $d(\rho)$, and $a(H) \in \mathbb{Q}$ are chosen to satisfy $\rho = \sum a(H)1_H^G$, then the methods above give

$$\pm \frac{c(\rho)}{2^{\rho(1-\gamma)/2}[L_\rho : \lambda M_\rho]^{1/d(\rho)}} = \prod_H \left\{\frac{h(H)}{n(H)w(H)[\overline{H\overline{U}} : HM]}\right\}^{a(H)}$$

where $L_\rho = L \cap e_\rho CL$ and $M_\rho = M \cap e_\rho CM$ for the central idempotent e_ρ of $\mathbb{Q}[G]$ corresponding to ρ. In [8] Stark derives essentially the same formula by generalizing the methods of Brauer.

4. Change of ground field. It remains to interpret Brauer's theorem in terms of the Galois group $\mathscr{G}$ of a relative normal extension $\mathscr{K}/\mathscr{k}$ within $K/\mathbb{Q}$. $\mathscr{H}$ will denote a subgroup of $\mathscr{G}$, $\mathscr{U}$ the unit group of $\mathscr{K}$, and $G(\mathscr{k})$ and $G(\mathscr{K})$ the Galois groups of $K/\mathscr{k}$ and $K/\mathscr{K}$. Then $\mathscr{L} = G(\mathscr{K})L$ is a $\mathbb{Z}[\mathscr{G}]$-module whose precise structure will be determined below.

THEOREM 4.1. *Suppose* $\sum a(\mathscr{H})1_{\mathscr{H}}^{\mathscr{G}} = 0$. *If the submodule* $\mathscr{M}$ of $\overline{\mathscr{U}}$ is $\mathbb{Z}[\mathscr{G}]$-isomorphic to $\mathscr{L}$ then

$$\prod h(\mathscr{H})^{a(\mathscr{H})} = \prod \left(n(\mathscr{H})w_2(\mathscr{H})[\overline{\mathscr{H}\mathscr{U}} : \mathscr{H}\mathscr{M}]\right)^{a(\mathscr{H})}$$

for $n(\mathscr{H}) = [\mathscr{H}\mathscr{K} : \mathscr{k}]$.

Proof. Put $H = G(\mathscr{K})\mathscr{H}$. Then $\mathbf{C}[\mathscr{G}]\tilde{\mathscr{H}}$ and $\mathbf{C}[G(\mathscr{k})]\tilde{H}$ are $\mathbf{C}[\mathscr{G}]$-isomorphic under $\tilde{\mathscr{H}} \leftrightarrow \tilde{H}$. So they have the same characters, i.e. $1_{\mathscr{H}}^{\mathscr{G}}(\mathscr{g}) = 1_H^{G(\mathscr{k})}(g)$ if $\mathscr{g} \in \mathscr{G}$ is the image of $g \in G(\mathscr{k})$. Hence the character relation $\sum a(H/G(\mathscr{K}))1_H^G = 0$ holds and Theorem 3.4 may be applied. Evidently $\overline{HU} = \overline{\mathscr{H}\mathscr{U}}$ and $HM = \mathscr{H}\mathscr{M}'$ for $H = G(\mathscr{K})\mathscr{H}$ and $\mathscr{M}' = G(\mathscr{K})M$. When these have been substituted Theorem 1.1 allows any $\mathscr{M} \cong \mathscr{L}$ to be chosen because $\mathscr{M}' \cong \mathscr{L}$, and Lemma 3.2 permits the new value of $n(\mathscr{H})$.

The generators HgC of $H\mathbb{Z}[G]\tilde{C}$ may be identified with the normalised infinite valuations

$$v_{HgC}(x) = \|g^{-1}x\|^f \qquad (x \in HK)$$

of HK where $f = |HgC|/|H|$ and $\|\ \|$ is the absolute value for the chosen embedding of K into $\mathbf{C}$. So the subgroup

$$\mathscr{C}_i = \left(g_iCg_i^{-1} \cap G(\mathscr{k})\right) / G(\mathscr{K})$$

of $\mathscr{G}$ which fixes $G(\mathscr{K})g_iC$ is the decomposition group in $\mathscr{K}/\mathscr{k}$ of the corresponding infinite prime. Thus the double coset decomposition

$$\tilde{G} = \sum_{i=1}^{r} G(\mathscr{k})g_iC$$

determines up to conjugacy a decomposition group $\mathscr{C}_i$ for each infinite prime of $\mathscr{k}$.

The exact sequence defining L restricts to

$$0 \to \mathbb{Z} \to G(\mathscr{K})\mathbb{Z}[G]\tilde{C} \to G(\mathscr{K})L \to 0\,.$$

This is also exact as fixing by a subgroup is a left exact functor and any pre-image of an element in $G(\mathscr{K})L$ is necessarily fixed by $G(\mathscr{K})$. However, $G(\mathscr{K})\mathbb{Z}[G]\tilde{C} \cong \oplus_{i=1}^{r}\mathbb{Z}[\mathscr{G}]\tilde{\mathscr{C}_i}$ under the $\mathbb{Z}[\mathscr{G}]$-map

$\sum x_i G(\mathscr{K}) g_i C \mapsto \oplus x_i \widetilde{\mathscr{C}_i}$ for $x_i \in \mathbb{Z}[\mathscr{G}]$. Hence

LEMMA 4.2. *If $\{\mathscr{C}_i\}$ is the set of decomposition groups for one prime divisor in $\mathscr{K}$ of each of the r infinite primes in $\mathscr{k}$ then $\mathscr{L}$ satisfies the exact sequence*

$$0 \to \mathbb{Z} \to \oplus_{i=1}^{r} \mathbb{Z}[\mathscr{G}]\widetilde{\mathscr{C}_i} \to \mathscr{L} \to 0$$

where $n \in \mathbb{Z} \mapsto n \oplus_i \widetilde{\mathscr{G}}$.

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