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Note

Intersection Numbers for Coherent Configurations and the Spectrum of a Graph

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It is shown that the intersection numbers of a coherent configuration are closely related to the determinant of a generic matrix for the corresponding adjacency algebra. This is important as both concepts provide isomorphism invariants for graphs.

The coherent configurations studied by Higman in [1] are of interest in graph theory because invariants for the isomorphism class of a graph are obtained in the intersection numbers of the unique minimal coherent configuration generated by the partition of $V\times V$ induced by the edge set E and the diagonal set V of the vertices of the graph.

If A is the adjacency matrix of a labelled, directed graph Γ with vertex set V, one constructs the smallest algebra $\mathscr{A}$ over $\mathbf{C}$ containing A, A^T and the characteristic matrices of sets $S \subset V\times V$ which are maximal with respect to

$$(i, j) \text{ and } (k, l) \in S \text{ implies } a_{ij} = a_{kl} \text{ for all } (a_{rs}) \in \mathscr{A} \tag{1}$$

These sets S form the partition of $V\times V$ which define a coherent configuration for Γ and their characteristic matrices (the 0,1-adjacency matrices of the graphs with edge sets S) generate $\mathscr{A}$ as a module over $\mathbf{C}$. Let $A_1, A_2, \ldots, A_r$ be these matrices. Suppose $X = \Sigma x_i A_i$ and $Y = \Sigma y_j A_j$ are two generic matrices of $\mathscr{A}$. Then the intersection numbers are defined by

$$XY = \Sigma\, a_{ijk} x_i y_j A_k \tag{2}$$

They are clearly independent of the vertex numbering and so are isomorphism class invariants. The same is true of the characteristic polynomial $\det(xI-A)$ and also of $\det X$, which certainly provides at least as much information as $\det(xI-A)$. The object of this paper is to show that the intersection numbers give a finer classification of graphs than does the spectrum or indeed $\det X$, and to investigate how much weaker is the

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classification using det X.

Suppose that A_i corresponds to $S_i \subset V{\times}V$. One may reasonably suppose that the numbers a_{ijk} are provided together with the partition of the sets S_i into diagonal and off-diagonal subsets. From (2) if S_d is a diagonal class with corresponding vertices V_d, then $S \cap V_d{\times}V_d = S_i$ or $\varnothing$ according as $a_{idi} = 1 = a_{dii}$ or not. Let $J_d = \Sigma' A_i$ where the sum is restricted to those i with $S_i \subset V_d{\times}V_d$. This is the matrix with 1's in the block $V_d{\times}V_d$ and 0's elsewhere. Hence $J_d^{\,2} = |S_d| J_d$. Equating coefficients of A_d using (2) yields the order of S_d as $\Sigma' a_{ijd}$, where i, j satisfy S_i, $S_j \subset V_d{\times}V_d$. Using (2) again one can inductively obtain the coefficients w_{dt} of each A_d for the powers $X^{\,t}$ ($t \le |V|$). Now trace(X_i) = $\Sigma_d w_{dt} |S_d|$ for the sum over the diagonal classes. Newton's formulae applied to the generic eigenvalues of X consequently yield det X.

THEOREM. *The intersection numbers determine the determinant of a generic matrix in the adjacency algebra.*

Conversely, assume that det X is given together with, as before, the partition into diagonal and off-diagonal sets. Let $\mathscr{B}$ be the algebra over **C** generated by $B_i = A_i^{\,T}$ for $1 \le i \le r$. This is the algebra of the graph $\Gamma^{\,T}$ with adjacency matrix $B = A^T$. Its intersection numbers satisfy $b_{ijk} = a_{jik}$ and its generic matrix is $Y = \Sigma x_i B_{\,i} = X^T$. Because det Y = det X, the determinant cannot distinguish a_{ijk} from a_{jik}. However, it will be shown that $a_{ijk}{+}a_{jik}$ is determined by det X.

Let $\{S_d \mid d{\in}D\}$ be the collection of diagonal sets with corresponding partition $V = \bigcup_{d\in D} V_d$. We will treat det X as a polynomial in the variables x_d ($d{\in}D$). The unique term z of highest degree has the form $\prod_{d\in D} x_d^{\;rd}$ and yields $|S_d| = r_d$ for $d{\in}D$. For $X = (x_{rs})$, the coefficient of $zx_i^{-1}x_j^{-1}$ for $i, j \in D$ is $-2^{-\delta}\Sigma_{r\in V_i}\Sigma_{s\in V}\, x_{rs}x_{sr} = -\tfrac{1}{2}\Sigma_k \delta_{ijk}\, |S_k|\, x_k\, x_{k'}$ where $S_{k'} = S_k^{\,T} = \{(r,s) \mid (s,r){\in}S_k\}$ defines k', δ = l or 0 according as $i = j$ or not, and δ_{ijk} = l or 0 according as $S_k \subset (V_i{\times}V_j) \cup (V_j{\times}V_i)$ or not. This pairs S_k with $S_k^{\,T}$, establishes when $S_k = S_k^{\,T}$ and so yields $|S_k|$ for all k.

The block of S_k is nearly determined: either $V_i{\times}V_j$ or $V_j{\times}V_i$ for i, j as above. Make an arbitrary choice of $S_k \subset V_i{\times}V_j$ for one such triple (i,j,k) with $i \ne j$. This fixes the precise block of all other classes as follows: The action of X on $\mathbf{C}^{|V|} = \mathbf{C}[v{\in}V]$ induces an action on the subspace $\mathbf{C}^r$ ($r = |D|$) with basis $\{\Sigma_{v\in V_d} v \mid d{\in}D\}$ which is described by the matrix $\overline{X}$ with entries $\overline{x}_{de} = |V_e|^{-1}\Sigma_{r\in V_d}\Sigma_{s\in V_e}\, x_{rs}$. Because the variables in each block are distinct, det X is an irreducible factor of det X with degree r. Set $x_s = 0$ if $S_s \cap V_d{\times}V_d = \varnothing$ for all $d{\in}D$, and $x_s = 1$ if $S_s \subset V_d{\times}V_d$ for some $d{\in}D$ but $s{\notin}D$. Then det X specialises to $\prod_{d\in D}(x_d-1)^{|Vd|-1}\,(x_d + |V_d| - 1)$ and det X can be recognised as that factor which specialises to $\prod_{d\in D}(x_d + |V_d| - 1)$. Now $S_s \subset V_i{\times}V \cup V{\times}V_j$ if, and only if,

$\det \overline{X}$ does not contain a term with the product $x_s x_k$ for i, j, k as chosen above. A couple of applications of this produces the block of every class S_s subject to the choice mentioned.

For the intersection numbers a_{uvw} where u or $v \in D$ and $w \notin D$ it is clear that $a_{iww} = a_{wjw} = 1$ for $S_w \subset V_i \times V_j$ and $a_{uvw} = 0$ otherwise; and if $k \in D$, then $a_{uu'k} = |S_u||S_k|^{-1}$ for $S_u \subset V_k \times V$ with $a_{uvk} = 0$ otherwise. Anyother intersection numbers contribute to the coefficient of $z(x_i x_j x_k)^{-1}$ in $\det X$ for some $i, j, k \in D$. This coefficient is obtained by looking at 3×3 submatrices of X which include the three diagonal elements x_i, x_j, x_k and equals

$$(4-p)!^{-1} \sum_{S_u \subset V_i \times V_j} \sum_{S_v \subset V_j \times V_k} \sum_{S_w \subset V_k \times V_i} |S_w|(a_{uvw'} x_u x_v x_w + a_{u'v'w} x_{u'} x_{v'} x_{w'}) \qquad (3)$$

where p is the number of distinct indices among i, j, k; the sums are restricted to off-diagonal classes (i.e., $u, v, w \notin D$); and $S_{t'} = S_t^T$ defines u', v', and w'. Symmetry in (3) for u, v, w ensures that

$$|S_u| a_{vwu'} = |S_v| a_{wuv'} = |S_w| a_{uvw'} \quad \text{and} \quad a_{uvw} = a_{u'v'w'} \qquad (4)$$

For $p>1$ the blocks $V_i \times V_j$, $V_j \times V_k$ and $V_k \times V_i$ are distinct and so $z(x_i x_j x_k)^{-1} x_u x_v x_w$ has coefficient $|S_w| a_{uvw'}$. This determines the intersection numbers except on diagonal blocks because those a_{uvw} not satisfying $S_u \subset V_i \times V_j$ $S_v \subset V_j \times V_k$ and $S_w \subset V_k \times V_i$ for some $i, j, k \in D$ with $u, v, w \notin D$ are necessarily zero.

Taking $p = 1$ in (3) and letting q be the number of distinct indices among u, v, w yields the coefficient

$$(4-q)!^{-1} \; |S_w|(a_{uvw'} + a_{vuw'}) \qquad (5)$$

for $z x_i^{-3} x_u x_v x_w$ by virtue of (4). The intersection numbers for the diagonal blocks obtained via (5) are the only ones that cannot be individually ascertained once the choice of block for one off-diagonal S_s is made. If the alternative choice holds so that all the classes S_s have been associated with the transpose of their correct block, then a_{uvw} and a_{vuw} have to be interchanged throughout. However, in either case $a_{uvw} + a_{vuw}$ is known.

THEOREM. *The determinant of a generic matrix in the adjacency algebra determines the sums $a_{ijk} + a_{jik}$ of pairs of intersection numbers.*

Let $\mathscr{A}_d$ be the algebra given by the block $V_d \times V_d$. Then $\mathscr{A}_d$ is commutative if, and only if, $a_{uvw} = a_{vuw}$ for all relevant u, v, w. In particular, this is the case when the rank or number of classes $S_s \subset V_d \times V_d$ is sufficiently small. Whenever $\mathscr{A}$ itself is commutative, the a_{uvw} can therefore all be found. Thus the intersection numbers and det X determine each other. For the important case when Γ is a regular (unlabelled) graph, $\mathscr{A}$ is simply the algebra generated by the adjacency matrix A and the all 1's matrix J. So $\mathscr{A}$ is commutative and det X yields the same class of graphs as do the intersection numbers.

REFERENCE

1. D. G. HIGMAN, Coherent configurations, *Geom. Dedicata* 4 (1975), 1-32.

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